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DETERMINING MOMENT OF INERTIA USING A THREE-WIRE PENDULUM: AN IN-DEPTH TUTORIAL

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Abstract: The moment of inertia, a fundamental property in physics and engineering, quantifies the resistance of a rigid body to changes in its rotational velocity. A larger moment of inertia signifies greater rotational stability. This concept holds significant importance in engineering applications, where mechanisms like engines employ sizable external wheels to enhance rotational steadiness. The magnitude of moment of inertia hinges on the body's mass, mass distribution, and axis of rotation. For objects with uniform mass distribution and regular shapes, calculating the moment of inertia around a fixed axis is straightforward, based on size and mass measurements. However, irregularly shaped objects with uneven mass distribution necessitate experimental measurements.

This paper explores the classical method of utilizing a three-wire pendulum to experimentally determine the moment of inertia. By measuring the torsion period and applying the principles of mechanical energy conservation, this approach offers several advantages, including the use of simple instruments, ease of operation, and high precision.

Keywords: Moment of Inertia, Rotational Stability, Three-Wire Pendulum, Mechanical Energy Conservation, Experimental Measurement

Introduction

1.1. Background

The moment of inertia is a measure of the rotational inertia of a rigid body. The greater the moment of inertia of a rigid body about an axis, the more difficult it is to change its angular velocity as it rotates about the axis that means it rotates more steadily. It has great significance in engineering, such as some engines have a large round wheel outside, is to increase the moment of inertia, so that the rotate speed is stable. The magnitude of moment of inertia is related to the mass of rigid body, mass distribution and the position of the rotation axis. For the object with uniform mass distribution and regular shape, the moment of inertia around the fixed axis can be calculated by measuring the size and mass, while for the object with uneven mass distribution and irregular shape, the moment of inertia only be measured by experiment. There are many experimental methods to measure the moment of inertia, and using three-wire pendulum to measure the moment of inertia of object is more classical. It is by measuring the torsion period and according to the conservation of mechanical energy to find the moment of inertia, and its advantages are: simple instrument, convenient operation, high precision.

1.2. Aim

The aim of this laboratory is to understand the moment of inertia of the object and its measurement method.

- Experimentally measure the moment of inertia of an object;
- The experimental moment of inertia compared with the theoretical value; Verify the parallel axis theorem for the moment of inertia.

2. Theory

2.1. Moment of inertia

For a body as shown in figure 1, who can be divided into many elemental parts, each of mass dm. The moment of inertia of the element about x-axis, y-axis and z-axis are, respectively. [1]

$I_{xx} = \int (y^2 + z^2)\,\mathrm{d}m$ (1)

$I_{yy} = \int (x^2 + z^2)\,\mathrm{d}m$ (2)

$I_{zz} = \int (x^2 + y^2)\,\mathrm{d}m$ (3)

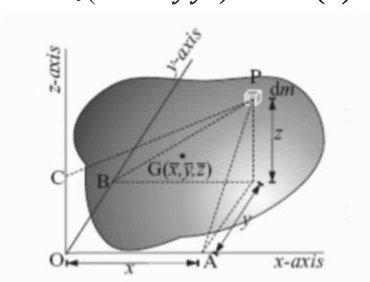
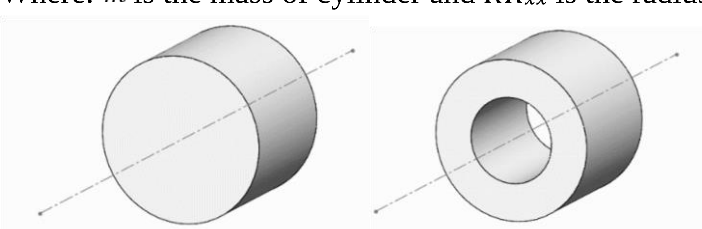


Figure 1: The moment of inertia of element [1]

As shown in figure 2(a), the moment of inertia of a cylinder about its central axis is

$I = \frac{1}{2} m R_x^{\,2}$ (4)

Where: m is the mass of cylinder and R_x is the radius of its base.



(a) *(b)*

Figure 2: Cylinder about central axis

As shown in figure 2(b), the moment of inertia of a ring about its central axis is

$I = \frac{m}{2}\left(R_1^{\,2} + R_2^{\,2}\right)$ (5)

Where: m is the mass of cylinder and R_1 and R_2 is the outer inter radius of its base.

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2.2. Theorem of parallel axes

It states that the moment of inertia of a body about any axis is equal to its moment of inertia about a parallel axis through its center of mass plus the product of the mass of the body and the square of the perpendicular distance between the axes. [1]

As shown in the figure 3, let the moment of inertia of a rigid body of mass M about the GG axis be II_{GGGG} (Point G is the center of mass). Then, the moment of inertia of the rigid body about the AB axis is

$II_{AAAA} = II_{GGGG} + MMyy_{cc}^2$ (6)

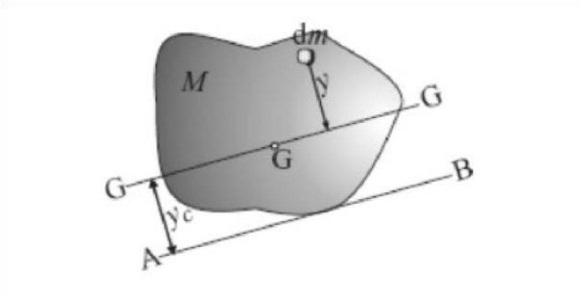

Figure 3: The rigid body rotates about the AB and GG axes [1]

2.3. Moment of inertia by using the three wire pendulum

As shown in figure 4, after the three-wire pendulum rotates angle θ, the position of the lower disk rises to h and its potential energy increases by EE_{pp} and its kinetic energy decreases by EE_{kk}.

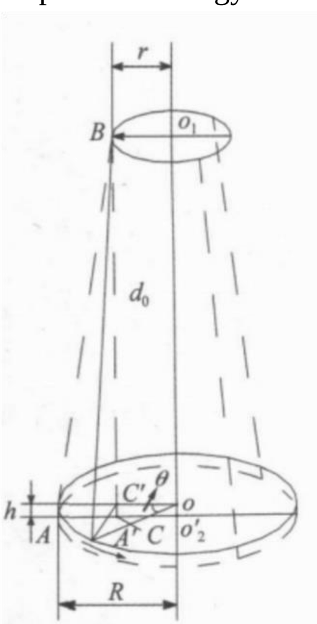

Figure 4: Sketch map of three-wire pendulum [2]

When the rotation angle θ is less than6°, according to the conservation of mechanical energy and Fourier series [2,3], then when the lower disk is not loaded, its moment of inertia II_0 is:

$$I_0 = \frac{m_0 g R r {T_0}^2}{4\pi^2 d_0} \quad (7)$$

Where: mm_0 is the mass of lower disk

RR, gg are the respective radius of the circle of the lower and upper three suspension points.
dd_0 is the vertical distance between the upper and lower disks at equilibrium.
TT_0 is the period time of simple mechanical vibration of the lower disk who is not loaded.
g is the acceleration of gravity (9.793mm/ss^2)
When the lower disk is loaded with the sample of mass m and moment of inertia I to be measured, having:

$$I + I_0 = \frac{(m+m_0)gRrT^2}{4\pi^2 d_0} \quad (8)$$

3. Experimental Method

3.1. Apparatus

- DH4601 The three-string torsional pendulum shown in figure 5 is the main apparatus of this experiment. In the experiment, the torsion pendulum movement of the lower disk around the central axis is achieved by rotating knob. Its oscillation period is measured by DHTC-3B Multifunction timer shown in figure 6 using cumulative amplification method.
- In the experiment, when the lower disk moves torsional pendulum, the light blocker passes through the photo receiver to intercept the light signal. At this moment, the multifunction timer records once, and finally stop after completing a pre-set number of times.
- Figure 7 shows ring sample, cylinder sample and bubble level from left to right. The bubble level is used to adjust the level of the upper and lower disks.
- Figure 8 shows the tape and Vernier caliper from up to down.

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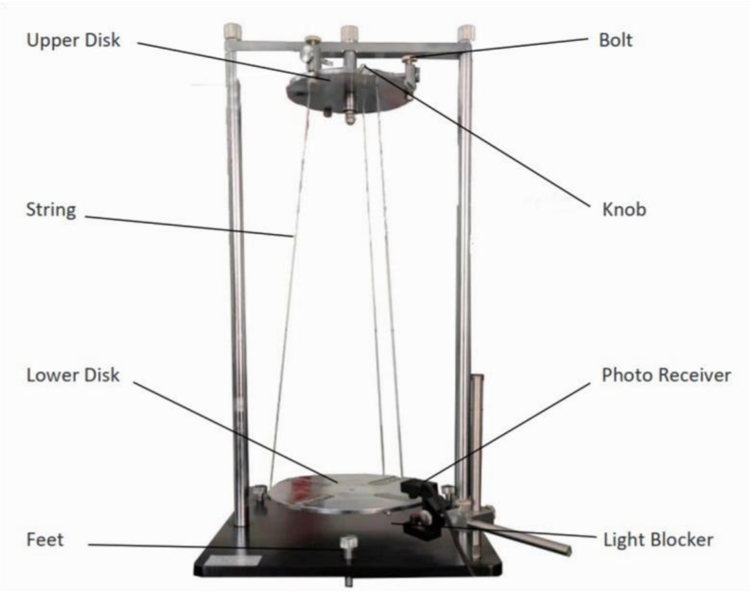

Figure 5: DH4601 The three-string torsional pendulum

Figure 6: DHTC-3B Multifunction timer

(a) (b) (c)

Figure 7: Ring (a), Cylinder (b) and bubble level (c)

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(a)- tape (b)- Vernier caliper Figure 8: Length measuring tool

3.2. Procedure

a) Measure the outer $2RR_1$ and inner diameters $2RR_2$ of the ring and the diameter $2RR_{xx}$ of the cylinder 5 times in different directions using vernier caliper. And measure the mass of the disk mm_0, the ring m and the two cylinders mm_1, mm_2. Record the data of the above measurements;

b) Place the bubble level on the upper disc and make the upper disc level by adjusting the feet, and then place the bubble level on the lower disc and make it level by adjusting the blot (adjusting bolts can change the length of the spring);

c) Measure the vertical distance dd_0 between the upper disk and lower disk using the tape. And measure the distance $\sqrt{3}r, \sqrt{3}R$ between the string holes of the upper and lower disks 5 times each.

Record the data of the above measurements;

d) Set the multifunction timer to count 30 times (15 periods);

e) Make the lower disk still, then rotate the upper disk to drive the lower disk twist. (In this process, three-line pendulum should not shake, and lower disk pendulum Angle should be less than 6 degrees). Start timing after the lower disk is balance. Record the time shown on the multifunction timer. Repeat the process five times;

f) Place the ring on the lower disk and make sure their centers coincide. Repeat step e);

g) Place symmetrically two cylinders on notch the of lower disk. Measure the distance $2x$ of two notches. Repeat step e).

4. Results

4.1. Raw Data

Table 1 shows the time of the disk in 15 periods when it is loading and non-loading.

Table 1: Cumulative method measurement cycle data record reference table

Time (s) required for 30 swings	Lower disk		Lower disk and ring		Lower disk and two cylinders	
	1	20.946	1	20.766	1	22.371
	2	20.972	2	20.750	2	22.388
	3	20.969	3	20.766	3	22.378
	4	20.973	4	20.800	4	22.379
	5	20.977	5	20.716	5	22.364
Period	TT_0 = 1.398		TT_1 = 1.384		TT_2 = 1.492	

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Table 2 shows the lengths required to be measured in the experiment.

Table 2: Reference table for data recording of multiple length measurements

Items / Times	Hanging hole spacing in upper plate, $\sqrt{3}r$ (cm)	Hanging hole spacing in lower plate, $\sqrt{3}RR$ (cm)	The ring		The cylinder diameter $2RR_{xx}$ (cm)	The distance between two cylinders $2x$ (cm)
			Outer diameter $2RR_1$ (cm)	Inner diameter $2RR_2$ (cm)		
1	7.75	15.90	15.081	10.100	2.981	17.10
2	7.80	16.00	15.080	10.101	2.981	17.10
3	7.70	16.10	15.082	10.100	2.983	17.09
4	7.72	16.05	15.084	10.098	2.980	17.11
5	7.78	15.95	15.083	10.100	2.980	17.10
Average	7.75	16.00	15.082	10.100	2.981	17.10

Other data used in the experiment are as follows (All are mean values of five measurements):

The mass of lower disk: $m_0 = 1058g$

The mass of ring: $m = 370.1g$

The mass of cylinder 1: $m_1 = 136.8g$

The mass of cylinder 2: $m_2 = 136.8g$

The vertical distance between the upper and lower disks when balancing: $d_0 = 43.50cm$

4.2. Sample Calculations

Theoretical value of Moment of inertia II_{tth1} of the ring is (using equation (5)):

$$I_{th1} = \frac{m}{2}\left(R_1{}^2 + R_2{}^2\right) = \frac{0.3701}{2} \times \left(\left(\frac{0.15082}{2}\right)^2 + \left(\frac{0.10100}{2}\right)^2\right) = 1.524 \times 10^{-3} kg \cdot m^2$$

Theoretical value of Moment of inertia II_{tth_cc} of a cylinder about its axis is (using equation (4)):

$$I_{th\_c} = \frac{1}{2} m_1 R_x{}^2 = \frac{1}{2} \times 0.1368 \times \left(\frac{0.02981}{2}\right)^2 = 1.520 \times 10^{-5} kg \cdot m^2$$

Theoretical value of Moment of inertia II_{tth2} of two cylinders with a distance of $2x$ about their center axis is (using equation (6)):

$$I_{th2} = 2\left(I_{th_c} + m_1 x^2\right) = 2 \times (1.520 \times 10^{-5} + 0.1368 \times \left(\frac{0.1710}{2}\right)^2) = 2.031 \times 10^{-3} kg \cdot m^2$$

Experimental value of Moment of inertia II_0 of the lower disk without load (using equation (7)):

$$I_0 = \frac{m_0 g R r T_0{}^2}{4\pi^2 d_0} = \frac{1.058 \times 9.793 \times 0.1600 \times 0.0775 \div 3 \times 1.398^2}{4 \times \pi^2 \times 0.4350} = 4.874 \times 10^{-3} kg \cdot m^2$$

Experimental value of Moment of inertia $II_1 + II_0$ of the lower disk and ring (using equation (8)):

$$I_1 + I_0 = \frac{(m+m_0) g R r T_1{}^2}{4\pi^2 d_0} = \frac{(0.3701 + 1.058) \times 9.793 \times 0.1600 \times 0.0775 \times 1.384^2}{4 \times \pi^2 \times 0.4350}$$
$$= 6.448 \times 10^{-3} kg \cdot m^2$$

Experimental value of Moment of inertia II_1 of the ring:

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$II_1 = 6.448 \times 10^{-3} - 4.874 \times 10^{-3} = 1.574 \times 10^{-3} kkgg \cdot mm^2$

Similarly, Experimental value of Moment of inertia II_2 of two cylinders with a distance of $2x$ can be obtained:

$II_2 = 2.113 \times 10^{-3} kkgg \cdot mm^2$ The relative error between theoretical value and experimental value calculation. [4]

$$e_{ring} = \left|\frac{I_1 - I_{th1}}{I_{th1}}\right| \times 100\% = \frac{1.574 \times 10^{-3} - 1.524 \times 10^{-3}}{1.524 \times 10^{-3}} \times 100\% = 3.28\%$$

$$e_{cylinder} = \left|\frac{I_2 - I_{th2}}{I_{th2}}\right| \times 100\% = \frac{2.113 \times 10^{-3} - 2.031 \times 10^{-3}}{2.031 \times 10^{-3}} \times 100\% = 3.88\%$$

Table 3 shows the theoretical and experimental value of moment of inertia of the ring and two cylinders with a distance of 2x, and the relative error.

Table 3: Theoretical value, experimental value and error

Moment of inertia	Theoretical value ($kkgg \cdot mm^2$)	Experimental value ($kkgg \cdot mm^2$)	Relative error (100%)
Ring	1.524×10^{-3}	1.574×10^{-3}	3.28%
Two cylinders	2.031×10^{-3}	2.113×10^{-3}	3.88%

5. Discussion

5.1. Comparison and Error Analysis

The relative error of the moment of inertia of the ring is 3.28% and the two cylinders, whose theoretical value was calculated by the parallel theory, is 3.88% as shown in table 3. By comparing experimental and theoretical values, it can be concluded within the permissible range of error that it is feasible to measure the moment of inertia of an object by three-wire pendulum and the parallel theory is verified, too.

 Formula (8) is approximated several times in its derivation. [5]

a) Approximation of Angle: sin $\theta\theta = \theta\theta$ that leads to an error of -0.05% ($\theta\theta = 3°$).

b) Neglect of the change of dd_0: leads to an error of -0.0011% ($\theta\theta = 3°$).

c) Neglect of air resistance: In fact, the resistance will increase the period and increase the result.

- The torsion of the lower disk will cause the whole three-wire pendulum to shake, which causes the mechanical energy of the lower disk and sample to decrease, resulting in a larger experimental result. The error can be reduced by adding weights to the base or avoiding excessive force when rotating the upper disk.
- Choose the applicable number of times to measure the period using cumulative amplification method. Too many times will lead to too much mechanical energy loss of lower disk, too few times will lead to period measurement misalignment.
- When measuring the moment of inertia of the ring, the rotating axis existed some distance apart from the central axis of the ring. According to the parallel theory, experimental result is on the large side.
- Swing off the axis of rotation: the rotation axis is not fixed when the lower disk is torsional.
- The lower disc must level in the experiment. If it is tilted, the theoretical axis of rotation of the sample will not coincide with the actual axis of rotation that means the experimental and theoretical moment of inertia are about different axes and this will result in no comparison between the experimental and theoretical values.

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5.2. Measurement of moment of inertia of irregular bodies by using three-wire pendulum

Measuring the moment of inertia of an irregular object with a three-wire pendulum is similar with the procedure in this experiment. But the irregular object need to ensure that its axis of rotation is fixed to the central axis of the three-wire pendulum, which is difficult to operate. This can be done by replacing the lower disc with a specific fixture.

5.3. The period of lower disk with or without sample

As shown in Table 1, the period of lower disk without sample is 1.398s, lower disk and ring is 1.384s and lower disk and two cylinders is 1.492s. It can be seen that the period may increase or decrease with the addition of sample.

In theory, equation (7) and (8) can obtain:

$$T_0^{\ 2} = \frac{m_0}{m_0 k_0^{\ 2}} \times \frac{gRr}{4\pi^2 d_0}$$

$$T^2 = \frac{m+m_0}{mk^2 + m_0 k_0^{\ 2}} \times \frac{gRr}{4\pi^2 d_0}$$

Where, kk_0 is the radius of gyration of lower disk.

k is the radius of gyration of sample.

Thus, $TT_0 = TT$, when $kk_0 = kk$.

$TT_0 < TT$, when $kk_0 < kk$.

$TT_0 > TT$, when $kk_0 > kk$.

According to the above analysis, the period is related to not only the mass of rigid body, but also mass distribution and shape (reflected in the radius of gyration).

6. Conclusion

6.1. Conclusion

- The method using three-wire pendulum to measure the moment of inertia of an object is reliable within the permissible range of error. And this is a basic method of measuring the moment of inertia of an irregular body.
- The parallel theory is verified. It states that the moment of inertia of an object about axis is the moment of inertia about axis passing through the centre of mass plus the distance between two axes squared multiply its mass.
- Cumulative amplification method is mastered, and it is often used to measure minute quantities.

6.2. Prospect

- Equation (7) can be improved by mathematical method to make the experimental results more accurate.
- Ref.5 provides a method for measuring the moment of inertia using a Laser Vibrometer. The accuracy of the system is better than 0.5%.

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AIP Conference Proceedings 1740, 090002