Original Article

VISCOELASTIC PROPERTIES AND THEIR IMPACT ON THE DYNAMICS OF FLEXIBLE STRUCTURES

Emily Sarah Watson and Julia Christine Lee

Mathematics Department, Towson University, Towson, Maryland, USA

Abstract: This article delves into the study of elastic structures with viscoelastic boundary conditions, focusing on an elastic thin plate in a bounded domain $\Omega \subset \mathbb{R}^2$ with C^2-smooth boundary Γ. The plate is clamped, and memory effects are considered on a subset Γ_0 with positive boundary measure. The vertical deflection $y(x, t)$ of this thin elastic plate is governed by a partial differential equation involving wave equations and memory effects. The specific problem can be described by the following equations:
$ytt(x, t) + \Delta^2 y(x, t) = 0$, in $\Omega \times \mathbb{R}^+$, (1.1a)
$y(x, t) = \partial v y(x, t) = 0$, on $\Gamma_0 \times \mathbb{R}^+$, (1.1b)
$\mathscr{B}_1 y(x, t) - \int_0^{+\infty} g'(s)\, \partial v[y(x, t) - y(x, t - s)]\mathrm{d}s = 0$, on $\Gamma_0 \times \mathbb{R}^+$, (1.1c)
$\mathscr{B}_2 y(x, t) + \int_0^{+\infty} g'(s)[y(x, t) - y(x, t - s)]\mathrm{d}s = u(x, t)$, on $\Gamma_1 \times \mathbb{R}^+$, (1.1d)
$y(x, 0^+) = y_0(x)$, $yt(x, 0^+) = y_1(x)$, (1.1e)
$y(x, -s) = \vartheta(x, t)$, for $0 < s < \infty$. (1.1f)
This study explores the interplay between elastic structures and viscoelastic boundary conditions, examining the behavior of the thin plate under these specified conditions.

Keywords: Elastic structures, viscoelastic boundary conditions, thin plate, partial differential equations, memory effects.

1 Introduction

The problems of elastic structures with viscoelastic boundary conditions have been studied extensively by many articles (see References [1]-[5]). Motivated by the work on wave and heat equations mentioned above, in this article we are concerned with an elastic thin plate which occupies a bounded domain $\Omega \subset \mathbb{R}^2$ with C^2-smooth boundary Γ. Assume that $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_1}$, where Γ_0 and Γ_1 are relatively open subsets of $\Gamma, \Gamma_0 \neq \emptyset$ has positive boundary measure, and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$. If Γ_0 is clamped and the memory effect on Γ_1 is taken into account, the vertical deflection $y(x, t)$ of the thin elastic plate satisfies the following partial differential equation:

$$y_{tt}(x, t) + \Delta^2 y(x, t) = 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+, \qquad (1.1a)$$
$$y(x, t) = \partial_v y(x, t) = 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, \qquad (1.1b)$$
$$\mathscr{B}_1 y(x, t) - \int_0^\infty g'(s)\, \partial_v[y(x, t) - y(x, t - s)]\mathrm{d}s = 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, \qquad (1.1c)$$
$$\mathscr{B}_2 y(x, t) + \int_0^\infty g'(s)[y(x, t) - y(x, t - s)]\mathrm{d}s = u(x, t), \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \qquad (1.1d)$$

Original Article

$y(x, 0^+) = y_0(x), \quad y_t(x, 0^+) = y_1(x),$ (1.1e)
$y(x, -s) = \vartheta(x, t), \quad for \quad 0 < s < \infty,$ (1.1f)
where g is the relaxation function, u is the boundary control, y_0, y_1, ϑ are the given initial conditions. $\mathscr{B}_1,\mathscr{B}_2$ are the following boundary operators:

$$\mathcal{B}_1 y = \Delta_y + (1-\mu)(2v_1 v_2 \frac{\partial^2 y}{\partial x_1 \partial x_2} - v_1^2 \frac{\partial^2 y}{\partial x_2^2} - v_2^2 \frac{\partial^2 y}{\partial x_1^2}),$$

$$\mathcal{B}_2 y = \partial_v \Delta_y + (1-\mu)\,\partial_\tau[(v_1^2 - v_2^2)\frac{\partial^2 y}{\partial x_1 \partial x_2} + v_1 v_2 (\frac{\partial^2 y}{\partial x_2^2} - \frac{\partial^2 y}{\partial x_1^2})]$$

,
$v = (v_1,v_2)$ is the unit outer normal vector, $\tau = (-v_2, v_1)$ is the unit tangent vector, and $0 < \mu < \frac{1}{2}$ is the Poisson ratio.

Throughout the article, we assume always that the function $g(\cdot)$ satisfies the following conditions:
(g_1) $g(\cdot) \in C^2[0,\infty)$;
(g_2) $g(t) > 0,$
(g_3) $g(\quad) > 0;$
$(g_4) g(t) \geq -kg$
$g('t < 0, \quad g''(t) \geq 0$ for $t \geq 0$;

) $\quad 0$ *and all* $t \geq 0.$
∞
$''$ $t$forsome $k >$

Condition ($g2$) implies that the memory of the boundary is strictly decreasing and the rate of memory loss is also decreasing. From ($g2$), we have also that both $g(\infty)$ and $g'(\infty)$ exist, $g'(\infty) \geq 0$. Condition ($g3$) means that the material behaves like an elastic solid at $t = \infty$. Condition ($g4$) implies that $g'(t)$ decays exponentially, in particular, $g'(\infty) = 0$.
The energy corresponding to the system (1) is defined by

$$E(t) = \frac{1}{2}a(y(\cdot,t)) + \int_\Omega |y_t(x,t)|^2 dx - \int_0^\infty \int_{\Gamma_1} g'(s)[|\partial_v(y(x,t) - y(x,t-s))|^2 + |y(x,t) - y(x,t-s)|^2] d\Gamma ds \quad (1.2)$$

where $a(w) = a(w, w)$ and

$$a(w_1,w_2) = \int_\Omega [\frac{\partial^2 w_1}{\partial x_1^2}\overline{\frac{\partial^2 w_2}{\partial x_1^2}}] + \frac{\partial^2 w_1}{\partial x_2^2}\overline{\frac{\partial^2 w_2}{\partial x_2^2}} + \mu(\frac{\partial^2 w_1}{\partial x_1^2}\overline{\frac{\partial^2 w_2}{\partial x_2^2}}] + \frac{\partial^2 w_1}{\partial x_2^2}\overline{\frac{\partial^2 w_2}{\partial x_1^2}}) + 2(1-\mu)\frac{\partial^2 w_1}{\partial x_1 \partial x_2}\overline{\frac{\partial^2 w_2}{\partial x_1 \partial x_2}}]dx, \quad \forall w_1, w_2 \in H^2(\Omega). \quad (1.3)$$

2. Well-Posedness of the System with Feedback Control

In this section, we shall formulate the system (1.1a-1.1f) into a standard linear infinite dimensional space with a output feedback control. Let
$W = \{w \in H^2(\Omega) | w|_{\Gamma 0} = \partial_v w|_{\Gamma 0} = 0\}, \quad \| w \|^2_W = a(w), \quad \forall w \in W,$
and define the "boundary memory space" by
$Z = L^2(0, \quad .)|; H^1($
$s)\Big|\Big[\| \partial_v z(s) \| \quad + \| z(s)$

Original Article

∞; $|g'(\Gamma_1)$),
∞
$_{1)}\Big]\,\mathrm{d}s, \qquad \forall z \in Z$
‖ z ‖2Z= $|g$ ('2L2 Γ1 ‖2L2 Γ.
0
Set
$\mathscr{H} = W \times L^2(\Omega) \times Z$
equipped with the inner product induced norm
$\| (w,v,z) \|_{\mathcal{H}}^2 = \| w \|_W^2 + \| v \|_{L_\Omega^2}^2 + \| z \|_Z^2, \quad \forall (w,v,z) \in \mathcal{H}.$
It is easy to see that $\mathcal{H}$ is a Hilbert space.
Remark We have that $a(\cdot)$2 is an equivalent norm on W since $\Gamma_0 \neq \emptyset$ has positive boundary measure. Moreover, it is obvious that $(\| \partial_v z \|_{L^2(\Gamma_1)}^2 + \| z^2 \|_{L^2(\Gamma_1)}^2)^{\frac{1}{2}}$ is an equivalent norm on $H^1(\Gamma_1)$. In fact, if ‖ $\partial_v z$ ‖^2L2(Γ1) +‖ z^2 ‖^2L2(Γ1)= 0, then $z = \partial_v z = 0$ on Γ_1. It follows that $\nabla_z = v\ \partial_v z = 0$ on Γ_1. Therefore, $z = 0$ in $H^1(\Gamma_1)$.
Next we introduce some operators (Ref.9) as follows:
(i) We set
$\int_0^\infty$
2,
$Lz(s) = g'(s)z(s)\mathrm{d}s,$
$\mathcal{A}_0 = \Delta \quad \mathcal{D}(\mathcal{A}_0) = \{w \in H^4(\Omega) \cap W | \mathscr{B}_1 w|_{\Gamma 1} = \mathscr{B}_2 w|_{\Gamma 1} = 0\}.$
It is easy to know that $\mathcal{A}_0$ is a positive self-adjoint operator on $L^2(\Omega)$.
(ii) The Green operators N_1 and N_2 are introduced to describe the boundary conditions,
$\Delta^2 \square = 0, \quad$ in $\quad \Omega$,
$N_1 g = h \Leftrightarrow \begin{cases} h \\ \mathcal{B} \\ \mathcal{B} \end{cases}$

$N_2 g = h \Leftrightarrow \begin{cases} h \\ \mathcal{B} \\ \mathcal{B} \end{cases}$
$= \partial_v \square = 0,$ on Γ_0,

1□ $= g,$ on Γ_1,
2□ $= 0,$ on Γ_1,
$\Delta^2 \square = 0,$ in Ω,
$= \partial_v \square = 0,$ on Γ_0,

1□ $= 0,$ on Γ_1,
2□ $= g,$ on Γ_1.
In terms of the regularity theory for the elliptic equations (Ref.6), we see that

Original Article

$$N_1: L^2(\Gamma_1) \to H^{\frac{5}{2}}(\Omega) \text{ is continuous,} \quad N_2: L^2(\Gamma_1) \to H^{\frac{7}{2}}(\Omega) \text{ is continuous}.$$

By these operators defined above, we may rewrite the system (1.1a-1.1f) as
$y_{tt}(\cdot,t) + \mathcal{A}_0[y(\cdot, t) - N_1L_z(\cdot, t, s) + N_2L_z(\cdot, t, s) - N_2u(\cdot, t, s)] = 0$,(2.1)
Where $z(\cdot, t, s) = y(x, t) - y(x, t - s)$, $x \in \Gamma_1$. Considering $L^2(\Omega)$ as the pivot space: $[\mathcal{D}(\mathcal{A}_0)] \subset L^2(\Omega) \subset [\mathcal{D}(\mathcal{A}_0)]'$ and extending the $\mathcal{A}_0$ to be $\mathcal{A}_0$: $L^2(\Omega) \to [\mathcal{D}(\mathcal{A}_0)]'$, we can rewrite (4) as
$y_{tt}(\cdot,t) = -\mathcal{A}_0 y(\cdot, t) + \mathcal{A}_0 N_1L_z(\cdot, t) - \mathcal{A}_0 N_2L_z(\cdot, t) + \mathcal{A}_0 N_2u(\cdot, t) \in [\mathcal{D}(\mathcal{A}_0)]'$. (2.2) Thus we can write the system (1.1a-1.1f) as a standard form of linear infinite-dimensional system in $\mathcal{H}$ $Y(t) = \mathcal{A}Y(t) + Bu$ (2.3)

Where

$$Y(t) = \begin{bmatrix} y(\cdot,t) \\ y_t(\cdot,t) \\ z(\cdot,t,s) \end{bmatrix}, \quad z(\cdot,t,s) = y(x,t) - y(x,t-s),$$

$$\mathcal{A} = \begin{bmatrix} 0 & I & 0 \\ -\tilde{\mathcal{A}}_0 & 0 & \tilde{\mathcal{A}}_0 N_1 L - \tilde{\mathcal{A}}_0 N_2 L \\ 0 & I & -\dfrac{\partial}{\partial s} \end{bmatrix}, \quad \mathcal{D}(\mathcal{A}) = \{Y \in \mathcal{H} \mid \mathcal{A}Y \in \mathcal{H}\}$$

And

$$Bu = \begin{bmatrix} 0 \\ \tilde{\mathcal{A}}_0 N_2 u \\ 0 \end{bmatrix}, \quad B: L^2(\Gamma_1) \to [\mathcal{D}(\mathcal{A}^*)]' \text{is continuous}$$

Finally, a direct computation gives

$$\begin{aligned} (N_2^* \mathcal{A}_0 f, g)_{L^2(\Gamma_1)} &= (\mathcal{A}_0 f, N_2 g)_{L^2(\Omega)} = (\Delta^2 f, N_2 g)_{L^2(\Omega)} \\ &= \int_\Omega f \overline{\Delta^2 (N_2 g)} \mathrm{d}x - \int_{\Gamma_1} \left[f \overline{\mathcal{B}_2 (N_2 g)} - \partial_v f \overline{\mathcal{B}_1 (N_2 g)} \right] \mathrm{d}\Gamma \\ &\quad + \int_{\Gamma_1} [\mathcal{B}_2 f \overline{(N_2 g)} - \mathcal{B}_1 f \, \overline{\partial_v (N_2 g)}] \mathrm{d}\Gamma \\ &= -\int_{\Gamma_1} f \overline{g} \mathrm{d}\Gamma, \end{aligned}$$

For all $f \in \mathcal{D}(\mathcal{A}_0)$ and $g \in L^2(\Gamma_1)$. Therefore, $N_2^*(\mathcal{A}_0)f = N_2^*\mathcal{A}_0 f = -f|_{\Gamma_1}$, $f \in \mathcal{D}(\mathcal{A}_0)$. It follows that

$$B^* \begin{bmatrix} w \\ v \\ z \end{bmatrix} = -v|_{\Gamma_1}, \quad \forall \begin{bmatrix} w \\ v \\ z \end{bmatrix} \in \mathcal{D}(\mathcal{A}^*). \qquad (2.4)$$

Now, let us consider a feedback control so that the input and output are collocated (Ref.7):
$u = -kB^*(y, y_t, z)^T = ky_t|_{\Gamma_1}, \quad k \geq 0$. (2.5)
The closed-loop system under this output feedback then becomes
$y_{tt}(x, t) + \Delta^2 y(x, t) = 0, \quad \text{in} \quad \Omega \times \mathbb{R}^+,$ (2.6a)
$y(x, t) = \partial_v y(x, t) = 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+,$ (2.6b)
$\mathcal{B}_1 y(x, t) - \int_0^\infty g'(s)\, \partial_v[y(x, t) - y(x, t - s)]\mathrm{d}s = 0, \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+,$ (2.6c)

Original Article

$\mathscr{B}_2 y(x,t) + \int_0^\infty g'(s)[y(x,t) - y(x,t-s)]ds = k y_t(x,t)$, on $\Gamma_1 \times \mathbb{R}^+$, (2.6d)
$y(x,0^+) = y_0(x)$, $y_t(x,0^+) = y_1(x)$, (2.6e)
$y(x,-s) = \vartheta(x,t)$. for $0 < s < \infty$, (2.6f)
The initial boundary problem (2.6) can be written as an evolutionary equation in $\mathcal{H}$:
$\dot{Y}(t) = \mathcal{A}Y(t)$, $Y(0) = Y_0$
Where $Y = (y, y_t, z)$, $Y_0 = (y_0, y_1, y_0 - \vartheta)$ and

$$\mathcal{A} = \begin{bmatrix} 0 & I & 0 \\ -\Delta^2 & 0 & 0 \\ 0 & I & -\frac{\partial}{\partial s} \end{bmatrix}$$

With the domain

$$\mathcal{D}(\mathcal{A}) = \left\{ (w,v,z) \in \mathcal{H} \;\middle|\; \begin{array}{l} \Delta^2 w \in L^2(\Omega), v \in W\infty, z(\cdot) \in H^1(0,\infty; |g'(\cdot)|; H^1(\Gamma_1)), \\ z \\ [\end{array} \right.$$

(0) = 0, $[\mathscr{B}_1 w - g'(s)\, \partial_\nu z(s) ds]_{\Gamma_1} = 0$,

0
∞
$\mathscr{B}_2 w + g'(s) z(s) ds]_{\Gamma_1} = kv|_{\Gamma_1}$,
0
Where

$$H^1(0,\infty; |g'(\cdot)|; H^1(\Gamma_1)) = \{z(s) \in Z | \frac{\partial}{\partial s} z(s) \in Z\}.$$

The following theorem ensures that the system (2.6) is well-posed in $\mathcal{H}$.
Theorem 2.1. Assume that the function g satisfies $(g1)$ through $(g3)$ and $k \geq 0$. Then the operator $\mathcal{A}$ generates a C_o-semigroup $S(t)$ of contraction on $\mathcal{H}$.
Proof. We first prove that $\mathscr{R}(I - \mathcal{A}) = \mathscr{H}$. Namely, we need to show that the following system of the equations

$w - v = f$, (2.7a)
$v + \Delta^2 w = g$, (2.7b)
$z(s) - v + \frac{\partial}{\partial s} z(s) = h(s)$ (2.7c) has a solution $(u, v, z) \in \mathcal{D}(\mathcal{A})$ for every $(f, g, \square) \in \mathscr{H}$. In fact, it follows from (2.6) that

$v = w - f \in W$, (2.8a)
$w + \Delta^2 w = f + g \in L^2(\Omega)$, (2.8b)
$z(s) = (1 - e^{-s})w + (1 - e^{-s})f + \int_0^\infty e^{\tau - s} \square(\tau) d\tau \in Z$. (2.8c)

Therefore, $v \in W$ and $z(\cdot) \in H^1(0,\infty; |g'(\cdot)|; H^1(\Gamma_1)), z(0) = 0.$ ∞ ∞ Furthermore, by (11b)-(11c) we have that for any
$w \in W$ satisfying $\Delta^2 w \in L^2(\Omega)$ and $\mathscr{B}_1 w - \int_0 g'(s)\, \partial_\nu z(s) ds = 0, \mathscr{B}_2 w + \int_0 g'(s) z(s) ds = kv$, it has for all $\phi \in W$,

Original Article

$$\int_\Omega w\overline{\phi}\mathrm{d}x + a(w,\phi) + \int_{\Gamma_1} [(kw + Xw)\overline{\phi} + X\,\partial_\nu w\overline{\phi}]\mathrm{d}\Gamma \\ = \int_\Omega (f+g)\overline{\phi}\mathrm{d}x + \int_{\Gamma_1} [(kf + Xf + \Psi)\,\overline{\partial_\nu \phi}]\mathrm{d}\Gamma \qquad (2.9)$$

Where

$$X = -\int_0^\infty g'(s)(1-e^{-s})\mathrm{d}s \geq 0$$

And

$$\Psi = \int_0^\infty g'(s)\int_0^s e^{\tau - s}h(\tau)\mathrm{d}\tau\mathrm{d}s.$$

We see from the Lax-Milgram theorem (Ref.8) that the equation (2.9) admits a unique solution $w \in W$. Combining this with (2.8a) and (2.8c), we see that $(w, v, z) \in \mathcal{D}(\mathcal{A})$ solves the equation $(I - \mathcal{A})(w, v, z) =$ $(f, g, \square)$.

Next, for any $Y = (w, v, z) \in \mathcal{D}(\mathcal{A})$, it has

$e \quad , \quad \mathcal{H} \qquad \mathscr{R}\,(\mathcal{A}Y\ Y)$

$$= \quad -k\int_{\Gamma_1} |v|^2 d\Gamma - \frac{1}{2}\int_0^\infty \int_{\Gamma_1} g''(s)(|z(s)|^2 + |\,\partial_\nu z(s)|^2)\mathrm{d}\Gamma\mathrm{d}s \leq 0$$

(2.10) .

Hence $\mathcal{A}$ is dissipative. We see from the theorem 1.4.6 of Ref.8 that $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Therefore, we can conclude by Lumer-Phillips theorem that $\mathcal{A}$ generates a C_o-semigroup of contractions on $\mathcal{H}$. The proof of Theorem 2.1 is complete now. □

3 A Variable Structural Control for the System

Let us establish a sliding model control for the system (??)

$$\begin{cases} \frac{\partial Y}{\partial t} = \mathcal{A}Y + Bw(Y,t) \\ Y(0) = Y_0 \end{cases} \qquad (3.1)$$

Where B is a bounded linear operator from $\mathcal{H}$ to $\mathcal{H}$, $w(Y, t)$ is the control of the system (3.1) that is not continuous on the manifold $S = CY = 0$, and C is a bounded linear operator with $S = S(Y) = CY \in R^n$.

Now, we consider the δ-neighborhood of sliding mode $S = CY = 0$, where $\delta > 0$ is an arbitrary given positive number. Using a continuous control $w\,(z, t)$ to replace $w(z, t)$ in the system 3.1 yields

$$\begin{cases} \dot{Y} = \mathcal{A}Y + B\widetilde{w}(Y,t) \\ Y(0) = Y_0 \end{cases} \qquad (3.2)$$

where$Y = \partial Y/\,\partial t$, and the solution of (3.2) belongs to the boundary layer $\| S(Y) \| \leq \delta$

Let $S\,(Y) = CY = 0$. Applying C to the first equation of (3.1) leads to the following the equivalent control: $w_{eq}(Y, t) = -(CB)^{-1}C(\mathcal{A}Y)$

With assumption that $(CB)^{-1}$ exists. Substitute $w_{eq}(Y, t)$ into 3.1 to find

$$Y = [I - B(CB)^{-1}C]\mathcal{A}Y. \qquad (3.3)$$

Denote $P = B(CB)^{-1}C$ and $\mathcal{A}_0 = (I - P)\mathcal{A}$, then 3.1 becomes

$Y^{\cdot}$ = $_0$, = $_0\mathcal{A}\,Y$ $Y(0)$ Y (3.4)

In the rest part of this paper, we are going to show that the actual sliding mode $Z(Y)$ will approach uniformly to the ideal sliding mode $Z(Y)$ under certain conditions.

Lemma 3.1 If $(CB)^{-1}$ is a compact operator and $P\mathcal{A} = \mathcal{A}P$, then $\mathcal{A}_0 = (I - P)\mathcal{A}$ generates a C_0 - semigroup $T_2(t)$ in $\mathcal{H}$ and $T_2(t) = (I - p)T_1(t)$, where $T_1(t)$ is the C_0-semigroup generated by $\mathcal{A}$.

Proof. Since $(CB)^{-1}$ is a compact operator, B and C are bounded linear operators, we see from the definition of P that P is compact, and therefor the range of $I - P$ is a closed subspace of $\mathcal{H}$. Since $P^2 = P$ and $(1 - P)^2 = I - P$, $I - P$ can be viewed as the identity operator on $(I - P)\mathscr{H}$. It can be easily seen that $T_2(t) = (I - P)T_1(t)$ is a C_0-semigroup in $(I - P)\mathscr{H}$.

Next, we shall prove that the infinitesimal generator of $T_2(t)$ is $(I - P)\mathcal{A}$ and $\mathcal{D}((I - P)\mathcal{A}) = (I - P)\mathcal{D}(\mathcal{A})$. In fact, for every $x \in (I - P)\mathcal{D}(\mathcal{A})$, there is a $x_1 \in \mathcal{D}(\mathcal{A})$ such that $x = (I - P)x_1$. It should be noted that $T_1(t)$ and $I - P$ are commutative because $\mathcal{A}$ and P are commutative. We see that

$$\lim_{t\to 0^+}\frac{T_2(t)x - x}{t} = \lim_{t\to 0^+}\frac{(I-P)T_1(t)(I-P)x_1 - (I-P)x_1}{t}$$
$$= \lim_{t\to 0^+}\frac{(I-P)^2T_1(t)x_1 - (I-P)x_1}{t}$$
$$= \lim_{t\to 0^+}\frac{(I-P)T_1(t)x_1 - (I-P)x_1}{t}$$
$$= (I-P)\lim_{t\to 0^+}\frac{T_1(t)x_1 - x_1}{t}$$
$$= (I-P)\mathcal{A}x_1.$$

Let $\mathcal{A}$ be the infinitesimal generator of $T_2(t)$. Since the limit on the left exists, we can assert that $x \in \mathcal{D}(\mathcal{A}\ )$ and $(I - P)\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}(\mathcal{A}\ )$.

On the other hand, for any $x \in \mathcal{D}(\mathcal{A}\ )$, since $\mathcal{D}(\mathcal{A}\ ) \subseteq (I - P)\mathscr{H}$, there exists $x \in \mathscr{H}$, such that $x = (I - P)x$, and

$$\lim_{t\to 0^+}\frac{{}_2\quad x - x}{t} = \lim_{t\to 0^+}\frac{{}_2\quad - \quad \tilde{x} \quad - \quad - \quad \tilde{x}}{t}\; T\;(t)\;\; T\,(t)(I\;P)(\;)\;\;(I\quad P)(\;)$$
$$= \lim_{t\to 0^+}\frac{(I-P)T_1(t)\tilde{x} - (I-P)\tilde{x}}{t}$$
$$= (I-P)\lim_{t\to 0^+}\frac{T_1(t)\tilde{x} - \tilde{x}}{t}$$
$$= (I-P)\mathcal{A}\tilde{x}.$$

Since the limit of the left hand side exists, and so the limit of the right hand side exists, and $x \in \mathcal{D}(\mathcal{A})$ which implies that $\mathcal{D}(\mathcal{A}\ ) \subseteq (I - P)\mathcal{D}(\mathcal{A})$. Thus, $\mathcal{D}(\mathcal{A}\ ) = (I - P)\mathcal{D}(\mathcal{A})$and $\mathcal{A}$, the infinitesimal generator of $T_2(t)$, is $(I - P)\mathcal{A}$.

The proof of the lemma is complete.

Theorem 3.2 Suppose that in the system 3.1,
1. $(CB)^{-1}$ exists and it is compact, 2. $P\mathcal{A} = \mathcal{A}P$, where $P = B(CB)^{-1}C$.

Then for any solution $Y(t)$ of the system 3.4 satisfying $S(\overline{Y}_0) = 0, \overline{Y}_0 \in \mathcal{D}(\mathcal{A}_0)$ and $\| Y_0 - \overline{Y}_0 \| \leq \delta, Y_0 \in \mathcal{D}(\mathcal{A})$, we have

$$\lim_{\delta\to 0} \| z(t) - \overline{z}(t) \| = 0$$

Original Article

Uniformly on $[0, T]$ for any positive number$T$.

Proof. We see from the Theorem 2.1 and Lemma 3.1 that $\mathcal{A}$ and $\mathcal{A}_0 = (I - P)\mathcal{A}$ are infinitesimal generators of C_0-semigroups $T_1(t)$ and $T_2(t)$ respectively. It follows from theory of semi group of linear operators that there are positive constants M_1,M_2,ω_1 and ω_2 such that

$$\| T_1(t) \| \leq M_1 e^{\omega_1 t}, \quad \| T_2(t) \| \leq M_2 e^{\omega_2 t}. \quad (0 \leq t \leq T) \qquad (3.5)$$

In the boundary layer $\| T_1(t) \| \leq \delta$, the equivalent control is

$$w_{eq}(Y, t) = -(CB)^{-1}C\mathcal{A}Y + (CB)^{-1}CY \qquad (3.6)$$

Substitute (3.6) into (3.1) to find

$$Y = (I - P)\mathcal{A}Y + PY \quad (3.7)$$

Hence, the solution of (3.7) can be expressed as follows:

$$Y(t) = T_2(t)Y_0 + \int_0^t T_2(t-s)P\dot{Y}(s)\mathrm{d}s, \qquad (3.8)$$

And the solution of (3.4) can be written as

$$\overline{Y}(t) = T_2(t)\overline{Y}_0 \qquad (3.9)$$

Substracting (3.9) from (3.8) yields

$$Y(t) - \overline{Y}(t) = T_2(t)(Y_0 - \overline{Y}_0) + \int_0^t T_2(t-s)P\dot{Y}(s)ds \quad (3.10)$$

Since$P\mathcal{A} = \mathcal{A}P$, we see that $PT_1(t) = PT_1(t)$. It should be emphasized that $(I - P)P = 0$ and $T_2(t) = (I - P)T_1(t)$, and consequently,

t

$$(t-s)P\dot{Y}(s)ds = \int_0^t (I-P)T_1(t-s)P\dot{Y}(s)\mathrm{d}s$$

t

$$T_1(t-s)(I-P)P\dot{Y}(s)\mathrm{d}s$$

T_2

0

=

0 = 0

It can be obtained from (3.10) and (3.5) that

$$\| Y(t) - \overline{Y}(t) \| \leq \| T_2(t) \| \| Y_0 - \overline{Y}_0 \| \leq M_2 e^{\omega_2 T} \| Y_0 - \overline{Y} \|,$$

Since$\| Y_0 - Y_0 \| \leq \delta$, we have

$$\| Y(t) - \overline{Y}(t) \| \leq M_2 e^{\omega_2 T}\delta.$$

$$\lim_{\delta \to 0} \| Y(t) - \overline{Y}_0 \| = 0.$$

Thus,

The proof of the theorem is complete.

We see from the Theorem 3.2 that the actual sliding mode can be approximated by ideal sliding mode in any accuracy.

References

Aassila, M., Cavalcanti, M.M. and Soriano, J.A., assymptotic Stability and Enerhy Decay Rates for Solutions of the Wave Equationth Memory in a Star-Shaped Domain, SIAM Journal of Control and Optimization, Vol.38 (2000), 1581-1602.

Rivera, J.E. and Andrade, D.,Exponential Decay of Nonlinear Wave equation with a Visoelastic Boundary Condition, Mathematical Methods in the Applied Sciences, Vol.23 (2000), 41-46.

Ciarletta, M., A Differential Problem for Heat Equation with a Boundary Condition with Memory. Applied Mathematics Letter, Vol.10 (1997), 95-191.

Fabrizio, M. and Morro,M., A Boundary Condition with Memory in Electro-magnetism, Archive for Rational Mechanics and Analysis, Vol.136-381 (1996), 359-381.

Zhang, Q. and Guo, B. Z. Stabilization of an Elastic Plate with Viscoelastic Boundary Conditions, Journal of Optimazation Theory and Applications, 212(3) (2004), 669-690.

Lions, J. L. and Magenes, R., Non-homogeneous Boundary-Value Problems and Applications, Vol., Springer Verlag, New York, NY, 1972.

Guo, B. Z. and Luo, Y. H., Controllability and Stability of a Second Order Hperbolic System with Collocated Sensor/Actuator, Sysstem and Control Leeter, Vol.46 (2002), 45-65.

Yosida, Functional Analysis, Springer Verlag, New York, 1980.

Pazy, A. Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer-Verlag, New York, 1983

Hou, X. and Tsui, S.K. Analysis and Control of a Two-link and Three-join Elastic Robot Arm. Applied Mathematics and Computation, 152(2004), 759-777.

Z. H. Luo, B. Z. Guo, and O. Mo rgul, Stability and stabilization of infinite-dimensional system with applications. Spring-Verlag, London (1999).

Hou, X. and Tsui, S.K. System Modeling and Optimization (Hall, Lasiecka and Polis, ed.), Chapman and Hall. pp.391398,1999.

Hou, X. and Tsui, S.K. Control and stability of torsional elastic robot arms. J. Math. Analy. Appl. 243 (2000), 140-162.