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DEVELOPING EFFECTIVE REDUNDANCY ALLOCATION FRAMEWORKS TO BOOST SYSTEM RELIABILITY

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Abstract: NASA defines redundancy as the use of multiple independent means to achieve a task, a critical aspect of ensuring mission success and system reliability. This concept was exemplified during NASA's Apollo 10 mission, where the availability of redundant systems allowed the mission to continue when faced with technical challenges. However, in the design of complex systems, achieving high levels of redundancy to meet stringent reliability goals can introduce challenges related to cost, weight, and size constraints. Balancing these trade-offs can be a complex task, as optimization models for redundancy allocation may be challenging to solve, as noted by Chern in 1992. Nevertheless, Chern also identified specific redundancy allocation models that can be solved efficiently within polynomial time. This research explores the optimization of redundancy allocation in complex systems, aiming to strike a balance between reliability goals and resource constraints. By leveraging the models identified by Chern, this study offers insights into effective redundancy allocation strategies to enhance system reliability while considering practical limitations.

Keywords: redundancy allocation, reliability, optimization models, complex systems, resource constraints.

1. Introduction

NASA (2018) defines redundancy as the "use of more than one independent means to accomplish a given task." Sullivan (1969) described for readers of The New York Times how redundancy sustained NASA‟s Apollo 10 mission. When the spacecraft lost the use of a fuel cell, two additional fuel cells were available to provide electrical power and the mission continued. The trade-off in the design of a complex system is that stringent reliability goals may require high levels of redundancy while the added cost, weight or size may be inconsistent with the system‟s purpose or infeasible given the system‟s resources. Chern (1992) has shown that optimization models of such trade-offs may be difficult to solve. At the same time, he identified models of redundancy allocation that can be solved in polynomial time.

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$$\min \quad \sum_{i=1}^{n} c_i x_i \qquad (1)$$

$$\text{s.t.} \quad \prod_{i=1}^{n} (1-\rho^{x_i}) \geq R$$

$$x_i \geq 1, \quad x_i \text{ integral}, \quad i = 1, \ldots, n.$$

In the context of system design, this model represents a series of n independent subsystems, all built from identical components. The integer n is greater than 1. The parameter ρ represents the failure probability of the individual components; the objective coefficients $c_1, c_2, \ldots, c_n$ are positive integers. The integer variable x_i represents the number of independent components arranged in parallel in the i-th subsystem. This subsystem fails only if all components fail, so its reliability is equal to $(1-\rho^{x_i})$. Since the independent subsystems are arranged in series, the product $\prod_{i=1}^{n}(1-\rho^{x_i})$ represents the reliability of the entire system (Durivage, 2017). The parameter R represents the reliability required of this system. If $(1-\rho)^n \geq R$, then no redundancy is required. So, the parameters ρ and R are rational numbers in $(0, 1)$ for which $(1-\rho)^n < R$.

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Rice, Cassady and Wise (1999) have argued that special cases of redundancy allocation models may be solved relatively easily and that useful insights may be obtained from these solutions. This paper deals with a special case of Model (1) for which the search for an optimal solution can be limited to a relatively small ($O(\log_2(n))$ set of feasible solutions. Each candidate for the optimal solution can be represented by only three integers, and the size of each of these integers is($O(\log_2(n))$. This special case, called Model (2), has the constraints of Model (1) and the objective

$$\min \quad cx_1 + \sum_{i=2}^{n} x_i. \tag{2}$$

A feasible solution of Model (2) is an n-vector of positive integers that satisfies the reliability constraint. An optimal solution, $(x_1^*, x_2^*, \ldots, x_n^*)$, is a feasible solution for which $cx_1^* + \sum_{i=2}^n x_i^* \leq cx_1 + \sum_{i=2}^n x_i$ when $(x_1, x_2, \ldots, x_n)$ is feasible. The optimal objective value $cx_1^* + \sum_{i=2}^n x_i^*$ certainly increases as the coefficient c increases. However, it is possible to establish an upper bound on the coefficient c beyond which the coordinates of an optimal solution of Model (2) do not respond to changes in this coefficient.

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The next section deals with related work. Sections 3 and 4 present a characterization of a finite set that contains an optimal solution and a bound on the size of this set. Section 5 presents an bound on the coefficient c beyond which only the objective value, but not the coordinates, of an optimal solution of Model (2) responds to changes in the coefficient. Section 6 provides examples, discussion and conclusions.

2. Related Work

Reliability is a critical factor in the design of engineering systems, and active redundancy, used in Models (1) and (2), is only one of many ways in which redundant components can be configured to sustain the performance of a subsystem. Birolini (2017) and Elsayed (2021) give detailed accounts of how to design and test for conformance to reliability requirements; they also describe and analyze varied configurations of redundant components. Systems engineered for high reliability are often operated by people who provide critical services to their communities. Roe and Schulman (2016) study how management, regulation and political leadership can improve the reliability of interconnected infrastructures. They obsesrve that experienced operators of these infrastructures have valuable information about the need for redundancy in plans to recover from unexpected events and about problems caused by lack of redundancy when a system is used in circumstances not considered in its design. Rueda and Pawlak (2004) offer a brief history of reliability theories, which include not only optimization, but also techniques and concepts from probability theory, statistics, stochastic processes and visual modeling methods. Tillman, Hwang and Kuo (1977) classified reliability optimization problems and reviewed the techniques of mathematical programming then available to solve these problems. They found that no single method was best suited to all problems and that the computing time and memory required to achieve exact solutions might be unrealistic in practice. Mohamed, Leemis and Ravindran (1992) classified optimization problems for redundancy allocation and reliability allocation according to the structure of the modeled system and whether or not components were repairable. They added heuristics to the list of optimization techniques and reported on computational experiments that compared optimization methods. Kuo and Prasad (2000) characterized the use of meta-heuristics, such as simulated annealing, genetic algorithms and tabu search, for redundancy allocation as possibly the most attractive development of the 1990‟s. Kuo and Wan (2007) reported on new optimization methods, such as ant colony algorithms, and new modeling opportunities, such as modeling the type of redundancy as a decision variable.

Coit and Zio (2019) discuss the prospect of improving the reliability of complex systems through upgrades developed by integrating optimization models with operational data about system performance. Moskowitz and McLean (1956) studied a variant of Model (1), in which the requirement that the values of decision variables are restricted to the positive integers is relaxed so that the values of the decision variables need only be positive. They obtained the optimal solution and proposed that rounding would provide an adequate, if not exact, solution of the discrete model.

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Chern (1992) proved that Model (1) can be solved in polynomial time, although the related model with parameters ρ1, ρ2, . . . , ρn in place of the single parameter ρ is NP-hard. Nmah (2011) studied the continuous relaxation of this NP-hard model and obtained an explicit representation of the unique optimal solution. Bhattacharya and Roychowdhury (2014) studied a related model in which additional parameters allow the reliabilities of the redundant components to differ from the reliabilities of the components considered part of the original design. For Model (2), Nmah (2016) constructed examples to show that the distance between optimal solutions of the discrete redundancy allocation model and its continuous relaxation could be arbitrarily large. Kaufmann, Grouchko and Cruon (1977) developed an algorithm to produce an optimal solution of Model (1) for the objective vector (1, 1, . . . , 1). Nmah (2017) developed a faster algorithm for the same problem.

3. Isolating an Optimal Solution

A first step in the solution of Model (2) is to establish bounds on the possible values of x_1^*, the first coordinate of an optimal solution.

Definition 1 *The positive integers L, $\hat{U}$, and U and are defined by*

$$L = \lfloor \log_2(1-R)/\log_2(\rho) \rfloor + 1 = \lfloor \ln(1-R)/\ln(\rho) \rfloor + 1$$

$$\hat{U} = \lceil \log_2(1-R^{1/n})/\log_2(\rho) \rceil = \lceil \ln(1-R^{1/n})/\ln(\rho) \rceil$$

$$U = \begin{cases} \hat{U}-1 & \text{if } (\hat{U}-1, \hat{U}, \hat{U}, \ldots, \hat{U}) \text{ is feasible,} \\ \hat{U} & \text{otherwise.} \end{cases}$$

Since $(1-\rho)^n < R, \hat{U} \geq 2$; so U is, indeed, a positive integer. For some combinations of R, ρ and n, it can happen that $L = U$. For example, consider $R = 0.99$, $\rho = 0.1$ and $n = 1, 2, \ldots, 90$. In the next section, it will be convenient to use base-2 logarithms to compute L and $\hat{U}$ and to use natural logarithms to determine the order of magnitude of $\hat{U}$.

Proposition 1 *Model (2) has an optimal solution. If $(x_1^*, x_2^*, \ldots, x_n^*)$ is an optimal solution, then $x_1^* \leq x_i^*$ for $2 \leq i \leq n$ and $L \leq x_1^* \leq U$.*

Proof. The vector in which each coordinate is equal to $\hat{U}$ is feasible and its objective vaue is equal to $(c+n-1)\hat{U}$. The set of feasible solutions for which the objective value is no greater than $(c+n-1)\hat{U}$ is finite and contains a feasible solution for which the objective value is minimal. Any such feasible solution is optimal.

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If the first coordinate of an optimal solution is not the smallest, then a feasible solution obtained by switching the first coordinate with a smaller coordinate would have a strictly smaller objective value because $c > 1$.

For an optimal solution, $(c+n-1)x_1^* \leq cx_1^* + \sum_{i=2}^{n} x_i^*$. Since the vector $(U, \hat{U}, \hat{U}, \ldots, \hat{U})$ is feasible, $(c+n-1)x_1^* \leq cU + (n-1)\hat{U}$. If $U = \hat{U}$, then $x_1^* \leq \hat{U}$. If $U = \hat{U} - 1$, $(c+n-1)x_1^* \leq (c+n-1)\hat{U} - c$. In this case, since x_1^* is an integer, $x_1^* \leq \hat{U} - 1$. Nmah (2015) proved that L is a lower bound for each coordinate of any feasible solution. ■

Among all feasible solutions for which $x_1 = x$, the candidates for optimal solutions of Model (2) are limited to optimal solutions of this model

$$\min \quad \sum_{i=2}^{n} x_i \tag{3}$$

$$\text{s.t.} \quad \prod_{i=2}^{n}(1-\rho^{x_i}) \geq R/(1-\rho^{x})$$

$$x_i \geq 1, \quad x_i \text{ integral}, \quad i = 2, \ldots, n.$$

The next definitions show how to construct an optimal solution of Model (3).

Definition 2 *For a positive integer, x, in the interval $[L, U]$, the function $R(x)$ is defined as $R(x) = R/(1-\rho^x)$. The function $\hat{u}(x)$ is defined as*

$$\hat{u}(x) = \lceil \log_2(1 - R^{1/(n-1)}(x))/\log_2(\rho) \rceil.$$

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If $n > 2$, the function $i(x)$ is defined as

$$i(x) = \max\{i = 0, 1, \ldots, n-2 : (1-\rho^{\hat{u}(x)-1})^i (1-\rho^{\hat{u}(x)})^{n-1-i} \geq R(x)\}.$$

Since $x \geq L, 0 < R(x) < 1$. The $n-1$-vector in which each coordinate is equal to $\hat{u}(x)$ is feasible for Model (3), so the set of indices that determine $i(x)$ includes $i = 0$.

Definition 3 *Let x be an integer in the interval $[L, U]$. If $n = 2$, then $y(x) = \hat{u}(x)$. If $n > 2$ and $i(x) = 0$, then the $n-1$-vector $y(x)$ is defined coordinate-wise by $y_i(x) = \hat{u}(x)$ for $2 \leq i \leq n$. If $n > 2$ and $i(x) > 0$, then the $n-1$-vector $y(x)$ is defined coordinate-wise by*

$$y_i(x) = \begin{cases} \hat{u}(x) - 1 & for \quad 2 \leq i \leq i(x) + 1 \\ \hat{u}(x) & for \quad i(x) + 2 \leq i \leq n. \end{cases}$$

Theorem 1 *The set $\{(x,y(x))$: x an integer in $[L, U]\}$ contains an optimal solution of Model (2).*

Proof. The vector $(x, y(x))$ is feasible for Model (2) when x is an integer in $[L, U]$.

If $n = 2$, then $y(x) \leq x_2$ when (x, x_2) is feasible for Model (2).

If $n > 2$, Nmah(2017) showed that the $n-1$ vector $y(x)$ is an optimal solution of Model (3) for the parameters $R(x)$ and ρ.

That the set $\{(x, y(x)) : x$ an integer in $[L, U]\}$ contains an optimal solution then follows from Proposition 1. ■

4 Establishing a Polylogarithmic Bound

The computational effort required to find an optimal solution depends on the number of integers in the interval $[L, U]$ and on the effort to determine the vector $y(x)$, given an integer x in $[L, U]$.

Proposition 2 $U \leq \hat{U} < 1 + \ln(R)/n\ln(\rho) - \ln(n)/\ln(\rho)$.

Proof. By definition, $U \leq \hat{U} < 1 + \ln(1 - R^{1/n})/\ln(\rho)$. From the mean value theorem, $(1 - R^{1/n}) > R^{1/n}\ln(1/R)/n$, so

$$\ln(1 - R^{1/n}) > \ln(R)/n + \ln(\ln(1/R)) - \ln(n)$$
$$\ln(1 - R^{1/n})/\ln(\rho) < \ln(R)/n\ln(\rho) - \ln(n)/\ln(\rho).$$

■

Theorem 2 *The size of the set* $\{(x,y(x))$*: x an integer in* $[L, U]\}$ *is* $O(\log_2(n))$.

Proof. The number of elements in the set is no greater than U. Proposition 2 shows that the integer U is $O(\log_2(n))$. ■

Proposition 3 *If x is an integer in* $[L, U]$, *then*

$$\hat{u}(x) < 1 + \ln(R(L))/(n-1)\ln(\rho) - \ln(n-1)/\ln(\rho). \quad (4)$$

Proof. If x is an integer in $[L, U]$, then $\hat{u}(x) \leq \hat{u}(L)$. Replace R with $R(L)$ and n with $n - 1$ in the inequalities of Proposition 2. ■

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Proposition 4 *If x is an integer in $[L, U]$, then the integer $i(x)$ can be computed in $2\lceil \log_2(n-1) \rceil$ steps.*

Proof. The configuration in which the subsystems numbered $2, 3, \ldots, n$ each consists of $\hat{u}(x)$ components conforms to the reliability constraint of (3). The configuration modelled by the vector $y(x)$ can be constructed by removing one component from as many of these subsystems as possible without violating the reliability constraint. The integer $i(x)$ is equal to the number of removed components, so $0 \leq i(x) \leq n-2$ because, by definition, the configuration in which the subsystems numbered $2, 3, \ldots, n$ each consists of $\hat{u}(x) - 1$ components is not feasible for Model (3).

The binary expansion of $i(x)$ can be constructed as follows:

Set $k_0 = 1$; as long as $2k_j < n-1$, set $k_{j+1} = 2k_j$; stop as soon as $2k_j \geq n-1$, and set $K = j+1$. Now, set $t = (1-\rho^{\hat{u}(x)-1})/(1-\rho^{\hat{u}(x)})$. Set $i_K(x) = 0$.

For $j = K-1, K-2, \ldots, 0$, if

$$t^{[2^j + i_{j+1}(x)]}(1-\rho^{\hat{u}(x)})^{n-1} \geq R(x),$$

set $i_j(x) = i_{j+1}(x) + 2^j$; otherwise, $i_j(x) = i_{j+1}(x)$. Then $i_0(x)$ belongs to the set of integers determining $i(x)$ and no integer in the set has a larger binary expansion, so $i_0(x) = i(x)$.

Since $2^{K-1} < n-1 \leq 2^K$, $K = \lceil \log_2(n-1) \rceil$ and the length of the iterative procedure is $2\lceil \log_2(n-1) \rceil$. ■

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The study of the sensitivity of an optimal solution of Model (2) to the objective coefficient c begins with a few observations about the function $\hat{u}(x)$. Throughout this section, it is assumed that $L < U$ so that the set of candidates for an optimal solution has more than one element.

Proposition 5 *For an integer, x, in $[L, U]$, the integer $\hat{u}(x)$ can be computed in $2\lceil\log_2(\hat{u}(x)\rceil$ steps.*

Proof. Modify the iterations of Proposition 4 as follows. Set $k_0 = 1$; as long as $(1 - \rho^{2k_j})^{n-1} < R(x)$, set $k_{j+1} = 2k_j$; set $K = j + 1$ as soon as the inequality fails. Then set $\hat{u}_K(x) = 2^K$. For $j = K-1, K-2, \ldots, 0$, set $\hat{u}_j(x) = \hat{u}_{j+1}(x) - 2^j$ if $(1 - \rho^{\hat{u}_{j+1}(x)-2^j})^{n-1} \geq R(x)$; set $\hat{u}_j(x) = \hat{u}_{j+1}(x)$ otherwise. Then $\hat{u}_0(x) = \hat{u}(x)$ and $K = 2\lceil\log_2(\hat{u}(x)\rceil$. ■

Theorem 3 *An optimal solution of Model (2) can be found in $O(\log_2^2(n))$ steps.*

Proof. The vector $(x, y(x))$ can be constructed from the three integers x, $\hat{u}(x)$ and $i(x)$. These integers also determine the objective value of $(x, y(x))$; in fact,

$$z(x) = cx + (n-1)\hat{u}(x) - i(x).$$

Theorem 2 shows that the number of candidates for the integer x is $O(\log_2(n))$. Propositions 3 and 4 show that the number of steps needed to compute the integers $\hat{u}(x)$ and $i(x)$ is $O(\log_2(n))$. Therefore, the overall effort is $O(\log_2^2(n))$. ■

5. Lack of Sensitivity of an Optimal Solution to Large Values of the Coefficient c

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Proposition 6 *If x is an integer in $[L, U]$, then $\hat{u}(x) \geq \hat{u}(U) = \hat{U}$. If x' is an integer in $[L, U]$ and $x' > x$, then $\hat{u}(x) \geq \hat{u}(x')$.*

Proof. From the definition of $\hat{u}(x)$,

$$(1-\rho^{x})(1-\rho^{\hat{u}(x)})^{n-1} \geq R.$$

Since $x \leq U$, $(1-\rho^{U}) \geq (1-\rho^{x})$, so

$$(1-\rho^{U})(1-\rho^{\hat{u}(x)})^{n-1} \geq R.$$

Then, by definition of $\hat{u}(U)$, it follows that $\hat{u}(x) \geq \hat{u}(U)$. For an integer x' in $[L, U]$ with $x' > x$, the same reasoning shows that $\hat{u}(x') \leq \hat{u}(x)$.

If $U = \hat{U} - 1$, then $(1-\rho^{U})(1-\rho^{\hat{U}})^{n-1} \geq R$ and then $\hat{u}(U) \leq \hat{U}$. But the definition of $\hat{U}$ implies that $R > (1-\rho^{\hat{U}-1})^{n}$. Then

$$R/(1-\rho^{U}) > (1-\rho^{\hat{U}-1})^{n-1}$$

so $\hat{u}(U) > \hat{U} - 1$. Since $\hat{u}(U)$ is an integer, it is equal to $\hat{U}$. If $U = \hat{U}$, the definition gives the inequalities $(1-\rho^{U})^{n} \geq R > (1-\rho^{U-1})(1-\rho^{U})^{n-1}$. Thus, $\hat{u}(U) \leq U$. But $(1-\rho^{U-1})(1-\rho^{U})^{n-1} > (1-\rho^{U-1})^{n-1}(1-\rho^{U})$, so $\hat{u}(U) > U - 1$ and, also in this case, $\hat{u}(U) = \hat{U}$. ■

Definition 4 *If x is an integer in $[L, U]$, then $s(x) = \sum_{i=2}^{n} y_i(x)$. The objective value, $z(x)$, of the vector $(x, y(x))$ can be written as $cx + s(x)$.*

Proposition 7 *If x is an integer in $[L, U]$, then $s(x) \geq s(U)$. If x' is in $[L, U]$ and $x' > x$, then $s(x) \geq s(x')$.*

Nmah(2017) showed that

$$\sum_{i=1}^{n} x_i \geq U + s(U)$$

for any feasible solution, $(x_1, x_2, \ldots, x_n)$, of Model (2). For the feasible solution $(x, y(x))$, this result implies $s(x) \geq (U - x) + s(U)$and so $s(x) \geq s(U)$. By definition,

$$(1 - \rho^x) \prod_{i=2}^{n} (1 - \rho^{y_i(x)}) \geq R.$$

If x' is an integer in $[L, U]$ and $x' > x$, then $(1-\rho^{x'}) > (1-\rho^x)$, so $(x', y(x))$ is feasible for Model 2. From the definition of $y(x')$, it follows that $s(x') \leq s(x)$.
■

Proposition 8 *If x is an integer in $[L, U)$, and $s(x) - s(U) - 1 < c$, then $z(x) \leq z(x')$ when x' is an integer in $(x, U]$.*

Proof. If x' is an integer in $(x, U]$, write x' as $x + \delta'$ where δ' is an integer and $1 \leq \delta' \leq U - x'$. If $z(x') < z(x)$, then

$$c(x + \delta') + s(x + \delta') \leq cx + s(x) - 1.$$

The term cx can be eliminated from each side of the inequality; from the previous proposition, $s(U) \leq s(x + \delta')$, so $c\delta' + s(U) \leq s(x) - 1$ and $c\delta' \leq s(x) - s(U) - 1$.

Since δ' is an integer, $\delta' \leq \lfloor (s(x) - s(U) - 1)/c \rfloor$. But the hypothesis of the proposition then shows that $\delta' \leq 0$, which is inconsistent with $\delta \geq 1$. ■

With these propositions, it is possible to show that as the objective coefficient c increases, the optimal vector(s) for Model (2) eventually fails to respond.

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Theorem 4 *If $s(L) - s(U) - 1 < c$, then $(L, y(L))$ is optimal for Model 2.*

Proof. Take $x = L$ in Proposition 8. ■

6. Examples and Discussion

The main result of this work is to limit the search for an optimal solution of Model (2) to a relatively small set ($O(\log_2(n))$ of candidates. The difference $U - L$ is the exact number of elements in the test set. Since $\hat{U} \geq U \geq \hat{U} - 1$ and since L does not depend on n, the growth of the set of candidates can be assessed from the growth of $\hat{U}$. Table 1 shows $\hat{U}$ as a function of the parameters ρ and n. For components of even moderate reliability, the set of candidates for an optimal allocation of redundancy is not large.

Table 1: $\hat{U}$ as a function of n and ρ for $R = 0.99$

n	2	4	8	16	32	64	128	256	512	1024
$\rho = 0.1$	3	3	3	4	4	4	5	5	5	6
$\rho = 0.5$	8	9	10	11	12	13	14	15	16	17
$\rho = 0.9$	51	57	64	70	77	84	90	97	103	110

The results of Sections 3 and 4 show that each candidate for an optimal solution can be represented by the three integers $x, \hat{u}(x)$ and $i(x)$; the size of each of these is $O(\log_2(n))$.

For some combinations of the parameters c, n, ρ and R, Model (2) may have multiple optimal solutions, including some not in the set of candidates identified in Section 3. For example, for $c = 2, n = 4, \rho = 0.9$ and $R = 0.99$, the test set of Section 3 includes four optimal solutions corresponding to $x = 51, 52, 53$ and 54. However, the permutations of optimal test vector $(52, 59, 59, 60)$ such as $(52, 59, 60, 59)$ and $(52, 60, 59, 59)$ are also optimal. If $x*$ determines an optimal solution for which $i(x*) > 0$, there are at least $\binom{n}{i(x*)}$ optimal solutions. In addition, Model (3) often has optimal solutions in which the difference between the largest and smallest coordinates is greater than 1; these, in turn, lead to solutions of Model (2) that are not included in the test set. So, in our example, the vector (52, 58, 60, 60) is optimal.

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The continuous relaxation of Model (2) is the optimization problem with the objective and reliability constraint of Model 2, but defined for variables that take positive, real values. One of the original approaches for solving Model (2) is to solve the continuous relaxation and use that solution to approximate an optimal solution of the discrete model (Moskowitz and McLean(1956)). Nmah (2015) showed that the optimal solution of the continuous relaxation of Model (2) has the form $(w*, f(w*), \ldots, f(w*))$ with $f(w*) = \ln(1 - R(w*)^{1/(n-1)}/\ln(\rho)$ and $w* > \ln(1 - R)/\ln(\rho)$. (Here, the function $R(\cdot)$ is the extension to the positive reals of the corresponding function in Definition 2 . The value $w*$ is determined by finding the unique positive root of a polynomial of degree n. When the objective coefficient c is large, $w* < L$ and $f(w*) > \hat{u}(L)$, so the feasible solution $(\lceil w* \rceil, \lceil f(w*) \rceil, \ldots, \lceil f(w*) \rceil)$ of Model 2 may be far from optimal (Nmah (2016). Theorem 4 provides an alternative for solving Model (2) for large values of c.

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